Semester One Examination 2011

Question/answer booklet

MATHEMATICS
3CMAT
Section One:
Calculator-free

Student Name: SOLUTIONS

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler, correction fluid/tape

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section One: Calculator–free **50 marks**

This section has **seven (7)** questions. Attempt **all** questions.

Question 1 **(8 marks)**

Differentiate the following, without simplifying.

(a) $y=\dfrac{2x-3}{(x-1)(x+1)}$ (3)

Solution
$y=\dfrac{2x-3}{x^2-1}$ ✓ $\dfrac{dy}{dx}=\dfrac{(x^2-1)2-(2x-3)2x}{(x^2-1)^2}$ ✓ ✓

(b) $y=(x+3)^4 e^{-5x}$ (2)

Solution
$\dfrac{dy}{dx}=e^{-5x}4(x+3)^3+(x+3)^4(-5e^{-5x})$

✓ ✓

(c) $y=5(x^2-4)^3$ where $u=x^2-4$ Use the chain rule notation $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ to differentiate. (3)

$y=5u^3 \quad \therefore \dfrac{dy}{du}=15u^2$ ✓ & $u=x^2-4 \Rightarrow \dfrac{du}{dx}=2x$ ✓

Hence $\dfrac{dy}{dx}=15u^2 \cdot 2x$

$=30u^2x$

but since $u=x^2-4$

$\Rightarrow \dfrac{dy}{dx}=30(x^2-4)^2x$

$=30x(x^2-4)^2$ ✓

Question 2 **(6 marks)**

(a) Evaluate $\int_1^2 (x+3)(x-1)\,dx$ (3)

Solution
$\int_1^2 x^2+2x-3\,dx = \left[\frac{x^3}{3}+\frac{2x^2}{2}-3x\right]_1^2$ ✓
$=\frac{2}{3}-\left(-\frac{5}{3}\right)$ ✓
$=\frac{7}{3}$ ✓

(b) Find $\int 6x^2\left(1-x^3\right)^5\,dx$ (3)

Solution
$\int 6x^2\left(1-x^3\right)^5\,dx = -2\int -3x^2\left(1-x^3\right)^5\,dx$ ✓
$=\frac{-2\left(1-x^3\right)^6}{6}+c$ ✓
$=-\frac{1}{3}\left(1-x^3\right)^6+c$ ✓

Question 3 **(6 marks)**

(a) A curve contains the point (1, 9) and the gradient of the curve at any point is given by $\frac{dy}{dx} = 6x - 6x^2$.

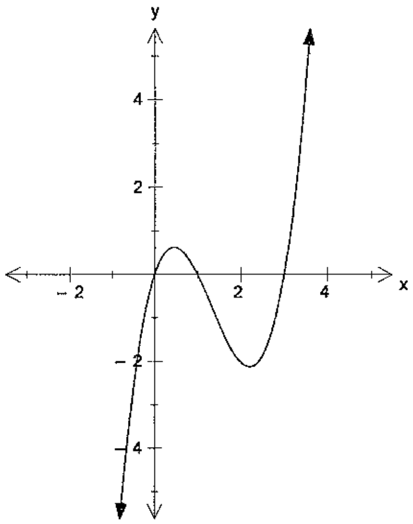
(i) Find the equation of the curve, 2 ~~(3)~~

Solution
$y = 3x^2 - 2x^3 + c$
Sub (1,9) into equation $9 = 3 - 2 + c$
$c = 8$
$\therefore$ equation is $y = 3x^2 - 2x^3 + 8$

(ii) State the number of solutions to the equation y = 8. 2 ~~(3)~~

Solution
Number of solutions is 2
$3x^2 - 2x^3 + 8 = 8$
$x^2(3 - 2x) = 0$
$x = 0$ or $x = \frac{3}{2}$

(b)



The area bounded by the curve $f(x) = x^3 - 4x^2 + 3x$ (drawn above) and the x-axis is calculated by integrating *f(x)* from *x* = 0 to *x* = 3 and this area is $\frac{37}{12}$ units^2.

However, on the CAS calculator, $\int_0^3 x^3 - 4x^2 + 3x \, dx$ results in an answer of $-\frac{9}{4}$.

Explain why the answers are different. (2)

Solution
The area from x =1 to x = 3 is below the x-axis, hence its integral is negative. Area is the sum of the absolute values of the 2 integrals resulting in $\frac{5}{12} + \frac{32}{12} = \frac{37}{12}$ while $\int_0^3 f(x)\, dx = \frac{5}{12} - \frac{32}{12} = \frac{-27}{12} = \frac{-9}{4}$

✓
✓

Question 4 **(4 marks)**

Variables x and y are related by the equation $y = \dfrac{2x-6}{x}$.

(i) Find an expression for $\dfrac{dy}{dx}$. (2)

Solution
$y = 2 - 6x^{-1}$ ✓ $\dfrac{dy}{dx} = 6x^{-2}$ or $\dfrac{dy}{dx} = \dfrac{6}{x^2}$ ✓

(ii) Hence, find an expression for the approximate increase in y as x increases from 4 to 4 + p, where p is small.

(2)

Solution
$\partial y = \dfrac{dy}{dx}.\partial x$ $= \dfrac{6}{x^2}\cdot p$ ✓ $= \dfrac{6}{4^2}\cdot p$ $= \dfrac{3p}{8}$ ✓

Question 5 **(~~5~~ 7 marks)**

(a) Simplify $\dfrac{2x+1}{x^2-1}-\dfrac{3}{x^2+x-2}$ (4)

Solution
$\dfrac{2x+1}{x^2-1}-\dfrac{3}{x^2+x-2}$ $=\dfrac{2x+1}{(x-1)(x+1)}-\dfrac{3}{(x+2)(x-1)}$ ✓ $=\dfrac{(2x+1)(x+2)-3(x+1)}{(x-1)(x+1)(x+2)}$ ✓ $=\dfrac{2x^2+4x+x+2-3x-3}{(x-1)(x+1)(x+2)}$ ✓ $=\dfrac{2x^2+2x-1}{(x-1)(x+1)(x+2)}$ ✓

(b) Simplify $\dfrac{5x^2-5}{x^2+4x-5}\div\dfrac{x^2-2x-3}{2x^2-18}$ (3)

$$=\frac{5(x^2-1)}{(x-1)(x+5)}\div\frac{(x-3)(x+1)}{2(x^2-9)} \quad ✓$$

$$=\frac{5\cancel{(x-1)}\cancel{(x+1)}}{\cancel{(x-1)}(x+5)}\times\frac{2\cancel{(x-3)}(x+3)}{\cancel{(x-3)}\cancel{(x+1)}} \quad ✓$$

$$=\frac{10(x+3)}{(x+5)} \quad ✓$$

Question 6 **(5 marks)**

A dice has two faces white, one blue and three red. It is thrown three times. What is the probability that

(i) a white face is uppermost at each throw. (1)

Solution
$\frac{2}{6}\times\frac{2}{6}\times\frac{2}{6}=\frac{1}{27}$

(ii) the same colour is uppermost at each throw. (2)

Solution
$\frac{2}{6}\times\frac{2}{6}\times\frac{2}{6}+\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}+\frac{3}{6}\times\frac{3}{6}\times\frac{3}{6}$ ✓
$=\frac{1}{27}+\frac{1}{216}+\frac{1}{8}$ $=\frac{8}{216}+\frac{1}{216}+\frac{27}{216}$
$=\frac{1}{6}$ ✓ $=\frac{36}{216}$

(iii) a different colour is uppermost at each throw. (2)

Solution
$\frac{2}{6}\times\frac{1}{6}\times\frac{3}{6}\times 3!$ ✓
$=\frac{1}{36}\times 3!$
$=\frac{1}{6}$ ✓

Question 7 **(4 marks)**

If $g(x)=\dfrac{5x+1}{3x+2}$ and $f(x)=\dfrac{7x+2}{6x+1}$, prove that $g(f(x))=f(g(x))$.

Solution
$g\left(\dfrac{7x+2}{6x+1}\right)=\dfrac{5\left(\dfrac{7x+2}{6x+1}\right)+1}{3\left(\dfrac{7x+2}{6x+1}\right)+2}$ ✓ $=\dfrac{5(7x+2)+1(6x+1)}{(6x+1)}\cdot\dfrac{(6x+1)}{3(7x+2)+2(6x+1)}$ $=\dfrac{35x+10+6x+1}{21x+6+12x+2}$ $=\dfrac{41x+11}{33x+8}$ ✓ $f\left(\dfrac{5x+1}{3x+2}\right)=\dfrac{7\left(\dfrac{5x+1}{3x+2}\right)+2}{6\left(\dfrac{5x+1}{3x+2}\right)+1}$ ✓ $=\dfrac{7(5x+1)+2(3x+2)}{6(5x+1)+1(3x+2)}$ $=\dfrac{41x+11}{33x+8}$ ✓   Hence, $g(f(x))=f(g(x))$

Section Two: Calculator–assumed **80 marks**

This section has **ten (13)** questions. Attempt **all** questions.

Question 8 **(7 marks)**

(a) The function $y = f(x) = e^{x(x^2-1)}$ is transformed to $y = 2e^{x(x^2-1)} + 1$.
Describe the transformation. (2)

Solution
The graph of f(x) is dilated by a factor of 2 parallel to the y-axis, followed by a translation of 1 unit in the y direction ✓ ✓

(b) Find the maximum and minimum values of the function $f(x) = 104 + 8x + \frac{288}{x}$, over the interval $1 \le x \le 7$. Show calculus techniques to gain full marks. (5)

Solution

$$f'(x) = 8 - \frac{288}{x^2}$$

$$f''(x) = \frac{576}{x^3} \quad ✓$$

$$8 - \frac{288}{x^2} = 0 \quad ✓$$

$$x^2 = 36$$

$x = \pm 6$ ignore $x = -6$, not in the given domain ✓

$f''(6) = +ve \Rightarrow \text{min}$

$f(6) = 200,\ f(1) = 400,\ f(7) = 201.14$ ✓

Maximum value of f(x) = 400
Minimum value of f(x) = 200 in the given interval ✓

Question 9 **(6 marks)**

(a) Greg tells the truth 3 out of 5 times and Ian tells a lie 4 out of 7 times. If they are asked about the same fact independently, what is the probability that they do not contradict each other? (3)

Solution
Greg and Ian do not contradict each other if both of them are telling the truth or telling a lie.
P(both telling the truth) $= \frac{3}{5} \times \frac{3}{7} = \frac{9}{35}$ ✓
P(both telling a lie) $= \frac{2}{5} \times \frac{4}{7} = \frac{8}{35}$ ✓
P(no contradiction) $= \frac{9}{35} + \frac{8}{35} = \frac{17}{35}$ ✓

(0.486)

(b) A shipment of 10 television sets contains 3 defective sets. In how many ways can the Boulton Hotel purchase 4 of these sets and receive at least 2 of the defective sets? (3)

Solution
2 defective + 3 defective
$= {}^3C_2\,{}^7C_2 + {}^3C_3\,{}^7C_1$ ✓✓
$= 63 + 7$
$= 70$ ✓

Question 10 **(4 marks)**

The variables y and t are related by the equation $y = ke^{-0.0231t}$ where k is a constant.

(a) When $t = 40$, $y = 28$, calculate the value of k. Express your answer to 3 significant figures. (1)

$28 = ke^{(-0.0231)40}$

Solution
$k = 70.5$ ✓

(b) When t = 50, calculate the value of

(i) y (1)

$y = 70.5\, e^{-0.0231(50)}$

Solution
y= 22.2 (to 3 sig fig) ✓

(ii) $\frac{dy}{dt}$ (2)

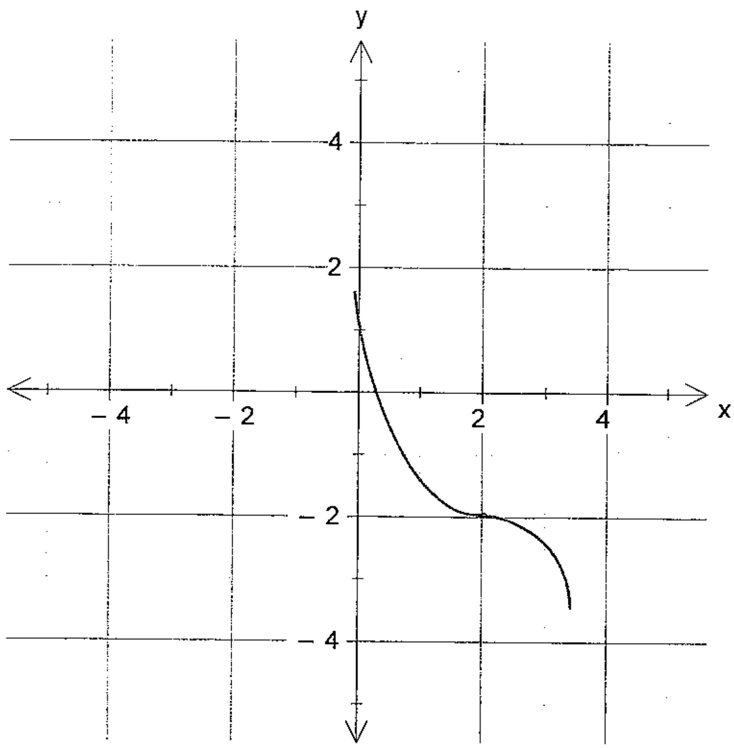
Solution
$\frac{dy}{dt} = -1.62855e^{-0.0231t}$ ✓
t= 50, $\frac{dy}{dt} = -1.62855e^{-0.0231\times50} = -0.513$ (to 3 sig fig) ✓

Question 11 **(9 marks)**

(a) Sketch a continuous curve for which

$f(0) = 1$
$f'(x) < 0$ and $f''(x) > 0$ for $0 < x < 2$
$f'(2) = 0$ and $f(2) = -2$
$f'(x) < 0$ and $f''(x) < 0$ for $x > 2$

(3)



Specific behaviours
√ correct shape
√ (2,-2)
√ slope for 0<x<2 and x >2

(b) Determine the domain and range of $f(g(x))$ given that $f(x) = \frac{12}{x+1}$ and $g(x) = \sqrt{x+1}$.

(3)

Solution
$f(g(x)) = \frac{12}{\sqrt{x+1}+1}$ ✓
$D_{fog} = \{x / x \geq -1, x \in R\}$ ✓
$R_{fog} = \cancel{\{y / y \geq 0, y \in R\}}$ ✓

$R_{fog} = \{y \in \mathbb{R},\ 0 < y \leq 12\}$

(c) Given that $f(x) = 2x + 3$ and $g(f(x)) = 4x^2 + 12x + 11$, find $g(x)$. (3)

$g(2x+3) = 4x^2+12x+11$ ✓

If $g(x) = x^2$, then

$g(2x+3) = (2x+3)^2 = 4x^2+12x+9$ ✓

but since require $gf(x) = 4x^2+12x+11$

$g(x)$ needs to be x^2+2 ✓

Alternative Solution

$g(x) = x^2+3x-7$

Solution
Let m = 2x + 3
$x = \frac{m-3}{2}$
$Now\ g(m) = 4\left(\frac{m-3}{2}\right)^2 + 12\left(\frac{m-3}{2}\right) + 11$
$= 4\left(\frac{m^2-6m+9}{4}\right) + 6(m-3) + 11$
$= m^2 - 6m + 9 + 6m - 18 + 11$
$= m^2 + 2$
$\therefore g(x) = x^2 + 2$

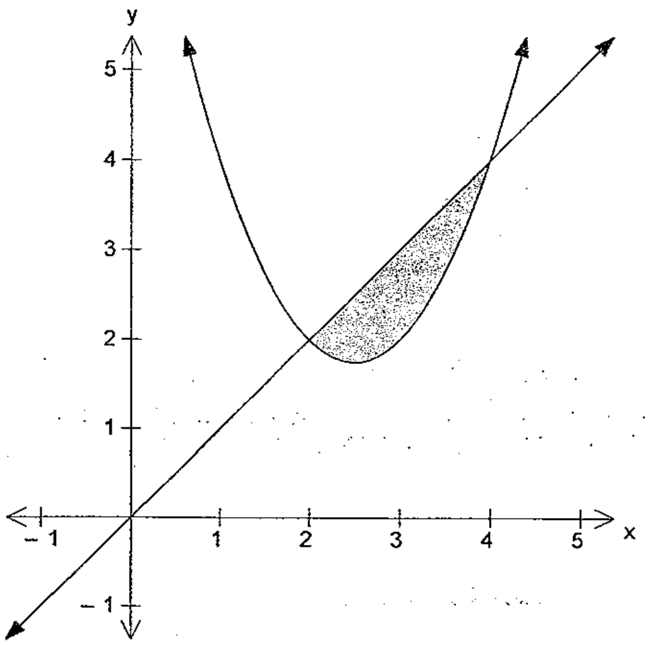
Question 12 **(8 marks)**

(a) In a binomial probability distribution, there are *n* trials and the probability of success for each trial is *p*. If the mean is 8 and the standard deviation is $\sqrt{4.8}$, find the values of *n* and *p*. (3)

Solution
np = 8
$\sqrt{np(1-p)} = \sqrt{4.8}$ ✓
$np(1-p) = 4.8$
$8(1-p) = 4.8$
$(1-p) = 0.6$
$p = 0.4$ ✓
$\therefore n = \frac{8}{0.4} = 20$ ✓
Hence, p = 0.4, n = 20

Question 13 **(4 marks)**

The line y = x intersects the curve $y = x^2 - 5x + 8$ at A(2, 2) and B(4, 4). The diagram shows the shaded region bounded by the line and the curve.
Find the area of the shaded region. Show full working.

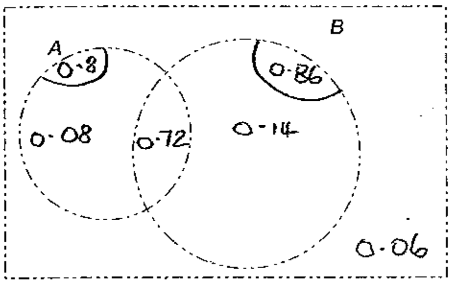


Solution
Shaded area = $\int_2^4 x\,dx - \int_2^4 x^2 - 5x + 8\,dx$ ✓ $= \left[\frac{x^2}{2}\right]_2^4 - \left[\frac{x^3}{3} - \frac{5x^2}{2} + 8x\right]_2^4$ ✓ $= 6 - \frac{14}{3}$ ✓ $= \frac{4}{3}$ unit^2 ✓

Question 14 **(9 marks)**

A Personal Identification Number (PIN) consists of 4 digits in order, each of which is one of the digits 0, 1, 2, ..., 9. Aimee has difficulty remembering her PIN. She tries to remember her PIN and writes down what she thinks it is. The probability that the first digit is correct is 0.8 and the probability that the second digit is correct is 0.86. The probability that the first two digits are both correct is 0.72.

By letting A = event that first digit is correct and
letting B = event that second digit is correct complete the Venn diagram and answer the following questions



✓ (1)

(a) Find the probability that the

(i) Second digit is correct given that the first digit is correct. (1)

$$\frac{0\cdot72}{0\cdot8} = 0\cdot9$$ ✓

(ii) First digit is correct and the second digit is incorrect. (1)

$$P(A \cap B') = 0\cdot08$$ ✓

(iii) First digit is incorrect and the second digit is correct. (1)

$$P(A' \cap B) = 0\cdot14$$ ✓

(iv) Second digit is incorrect given that the first digit is incorrect. (2)

$$P(B'/A') = \frac{P(B' \cap A')}{P(A')} \doteq \frac{0\cdot06}{0\cdot06+0\cdot14} \approx 0\cdot3$$ ✓ ✓

(b) Assuming the probability that all 4 digits are correct is 0.7. On 12 separate occasions Aimee writes down independently what she thinks is her PIN. Find the probability that the number of occasions on which all four digits are correct is less than 10.

(3)

Solution
$P(<10)$ = 1- P(10 occasions out of 12, 4 digits are correct) – P(11 out of 12, 4 digits are correct) – P(12 out of 12, 4 digits are correct) ✓ $=1-{}^{12}C_{10}(0.7)^{10}(0.3)^2-{}^{12}C_{11}(0.7)^{11}(0.3)^1-{}^{12}C_{12}(0.7)^{12}(0.3)^0$ ✓ $=1-0.1678-0.0712-0.0138$ $=0.7472$ ✓

Question 15 **(4 marks)**

Given that $\int_{2.5}^{k} e^{2x-5}\, dx = \frac{e-1}{2}$, find the value of *k*.

Solution
$\frac{1}{2}\int_{2.5}^{k} 2e^{2x-5}\, dx = \frac{e-1}{2}$ ✓ $\frac{1}{2}\left[e^{2x-5}\right]_{2.5}^{k} = \frac{e-1}{2}$ ✓ $\frac{1}{2}e^{2k-5} - \frac{1}{2} = \frac{e}{2} - \frac{1}{2}$ ✓ *i.e.* $2k-5=1$ $k = 3$ ✓

Question 16 **(5 marks)**

Consider the function $f(x) = 2x^3 + ax^2 + 3x + b$ where a and b are constants

(a) Find an expression for the gradient of the curve (1)

$f'(x) = 6x^2 + 2a + 3$ ✓

(b) Given that the tangents at A(0, b) and B(3,8) are parallel, find the values of a and b. (4)

$f'(0) = f'(3)$ ✓

$3 = 6(3)^2 + 2a + 3$

$3 = 54 + 2a + 3$

$3 = 57 + 2a$

$2a = -54$

$\therefore a = -27$ ✓

Given $f(3) = 8$

$\Rightarrow 8 = 2(3)^3 + (-27)(3)^2 + 3(3) + b$ ✓

$8 = 54 - 243 + 9 + b$

$\therefore b = 188$ ✓

Question 17 **(8 marks)**

(a) In the following table, x is a score in a game and P(X) is the probability of getting that score. The expected mean of the discrete probability distribution is 3.2. Find the values of m and n. (3)

x	1	2	3	4	5
$P(X = x)$	0.2	m	0.2	n	0.2

Solution
$0.2 + m + 0.2 + n + 0.2 = 1$ $m + n = 0.4$ - equation A ✓ $0.2 + 2m + 0.6 + 4m + 1 = 3.2$ ✓ $2m + 4n = 1.4$ $m + 2n = 0.7$ - equation B Solving A and B simultaneously, $m = 0.1$, $n = 0.3$ ✓

(b) The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that

(i) exactly 5 survive? (1)

Solution
B(5,15,0.4)□ $P(X\leq 5) - P(X\leq 4) = 0.4032\text{-}0.2173$=0.1859 ✓

(ii) at least 10 survive ? (2)

Solution
$P(X\geq 10) = 1 - P(X\leq 9) = 1 - Bin(9,15,0.4)$ ✓ $= 1 - 0.9662$ $= 0.0338$ ✓

(iii) from 3 to 8 survive? (2)

Solution
B(8,15,0.4) – B(2,15,0.4) ✓ = 0.9050 – 0.0271 = 0.8779 ✓

Question 18 **(5 marks)**

In the first five seconds of inflation, the relationship between the radius (r cm) and time (t sec) of a spherical party balloon are related by the formula

$$r = -t(t-10)$$

(a) Show that the relationship between volume (V cm^3) and time is given by (1)

$V = \frac{4\pi(10t-t^2)^3}{3}$

$r = (10t - t^2)$

$V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi(10t - t^2)^3$ ✓

which was to be shown

(b) Determine the exact volume of the balloon 3 seconds after inflation commenced. (1)

$V(3) = \frac{4}{3}\pi(10(3) - 3^2)^3$

$= \frac{4}{3}\pi(30-9)^3$ ✓

$= \frac{4}{3}\pi(21)^3 = 12348\pi \text{ cm}^3$

(c) Determine the approximate change in volume as t increases from 3 to 3.01 sec. (3)

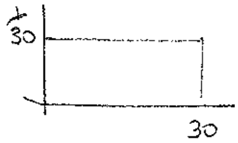
$\delta V = \frac{dV}{dt} \times \delta t$

$= 4\pi(10t - t^2)^2(10-2t) \times (0.01)$ ✓✓

$= 70.56\pi \text{ cm}^3$ ✓

Question 19 **(4 marks)**

Talia has calculated the arrival time of her mother to pick her up after school on any one day can be modelled by a uniform probability function with a maximum arrival time of 30 minutes. If this probability function proves a good estimate of future events, determine the probability on the next date, Talia will wait:



(a) 20 minutes (1)

$P(X=20) = 0$ (continuous data) ✓

(b) at least 25 minutes (1)

$5\left(\frac{1}{30}\right) = \frac{1}{6}$ ✓

(c) at least 25 minutes if she has to wait at least 10 minutes. (2)

$$P(X \geq 25 \mid X \geq 10) = \frac{P(X \geq 25)}{P(X \geq 10)} \checkmark$$

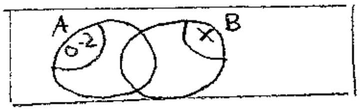
$$= \frac{\frac{1}{6}}{\frac{2}{3}}$$

$$= \frac{1}{4} \checkmark$$

Question 20 **(7 marks)**

Given only two events A and B are possible and $P(A) = 0.2$, $P(B) = x$ and $P(A \cup B) = p$:

(a) Find in terms of x,p and/or any numeric value, $P(A \cap B)$ (1)



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$p = 0{\cdot}2 + x - P(A \cap B)$

$\therefore P(A \cap B) = 0{\cdot}2 + x - p$ ✓

(b) If event A is a subset of event B determine a range of values for p. (1)

$0{\cdot}2 \le p \le x$

(c) If x=0.6, determine for what values of p are

(i) events A and B mutually exclusive? (2)

Mutually exclusive $\Rightarrow P(A \cup B) = P(A) + P(B)$ ✓

$p = 0{\cdot}2 + x$

$\therefore p = 0{\cdot}2 + 0{\cdot}6$

ie $p = 0{\cdot}8$ ✓

(ii) events A and B are independent? (3)

independent $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$ ✓

$0{\cdot}2 + x - p = 0{\cdot}2x$ ✓

$0{\cdot}2 + x - 0{\cdot}2x = p$

$\therefore p = 0{\cdot}2 + 0{\cdot}8x$ ie $p = 0{\cdot}2 + 0{\cdot}8(0{\cdot}6)$

$= 0{\cdot}2 + 0{\cdot}48$

END OF EXAM $= 0{\cdot}68$ ✓

Additional working space if needed

Question number(s):__________________